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# INDIVISIBILITY OF CENTRAL VALUES OF $L$ -FUNCTIONS FOR MODULAR FORMS

MASATAKA CHIDA

**ABSTRACT.** In this paper, we generalize works of Kohnen-Ono [7] and James-Ono [5] on indivisibility of (algebraic part of) central critical values of  $L$ -functions to higher weight modular forms.

## 1. INTRODUCTION

In this article, we show an indivisibility result on central critical values of  $L$ -functions associated to quadratic twists of modular forms using a method of Kohnen-Ono [7] and James-Ono [5].

Let  $f(z) = \sum_{n=1}^{\infty} a(n)q^n$  be a normalized newform of weight  $2k$  for  $\Gamma_0(N)$  with trivial character. For a fundamental discriminant  $D$  with  $(D, N)=1$ , we define the  $D$ -th quadratic twist of  $f$  by

$$f \otimes \chi_D = \sum_{n=1}^{\infty} a(n)\chi_D(n)q^n,$$

where  $\chi_D$  is the quadratic character corresponding to the quadratic extension  $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$ . Then  $f \otimes \chi_D$  is a newform of weight  $2k$  for  $\Gamma_0(D^2N)$ . Similarly, the  $D$ -th quadratic twist of the  $L$ -function  $L(f, s)$  is given by

$$L(f \otimes \chi_D, s) = \sum_{n=1}^{\infty} \frac{a(n)\chi_D(n)}{n^s}.$$

Let  $E$  be the number field generated by all Fourier coefficient of  $f$  and  $\mathbb{Q}$ . Then it is known that there exists a period  $\Omega_f \in \mathbb{C}^\times$  satisfying that  $\frac{L(f \otimes \chi_D, k)D_0^{k-1/2}}{\Omega_f}$  are integers in  $E$  for all fundamental discriminant  $D$  with  $\delta(f) \cdot D > 0$ , where  $\delta(f) \in \{\pm 1\}$  is the sign defined in Ono-Skinner [10, p. 655] and  $D_0$  is given by

$$D_0 = \begin{cases} |D| & \text{if } D \text{ is odd,} \\ |D|/4 & \text{if } D \text{ is even.} \end{cases}$$

We fix such a period  $\Omega_f$ .

For convenience, we denote

$$S(X) = \{D \in \mathbb{Z} \mid |D| < X, D : \text{fundamental discriminant}\},$$

and if functions  $f, g$  on  $\mathbb{R}$  satisfy that there is a positive constant  $c$  such that  $f(X) \geq c \cdot g(X)$  for sufficiently large  $X > 0$ , then we write  $f(X) \gg g(X)$ .

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**Theorem 1.1.** *Let  $f(z) = \sum_{n=1}^{\infty} a(n)q^n$  be a normalized newform of weight  $2k$  for  $\Gamma_0(N)$  with trivial character. Then, for all but finitely many primes  $\lambda$  of  $E$ , we have*

$$\#\left\{D \in S(X) \mid \delta(f) \cdot D > 0, \lambda \nmid D \text{ and } \frac{L(f \otimes \chi_D, k) D_0^{k-\frac{1}{2}}}{\Omega_f} \not\equiv 0 \pmod{\lambda}\right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

This result is a refinement of results of Bruinier [2] and Ono-Skinner [10]. The proof is based on a generalization of a method of Kohnen-Ono [7] and James-Ono [5]. In the above theorem, we do not assume that the Fourier coefficients of  $f$  belong to  $\mathbb{Z}$ , therefore it does not hold the surjectivity of the residual Galois representation associated to  $f$  for almost all places in general. This makes some technical difficulty on the proof. To solve this problem, we may use a result of Ribet [12] on the image of Galois representations associated to modular forms. This is an ingredient in our proof. In the last section, we also consider an indivisibility result on non-central critical values of  $L$ -functions for higher weight modular forms using congruences of modular form with different weights.

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## 2. MODULAR FORMS OF HALF-INTEGRAL WEIGHT

We denote the space of modular forms of weight  $k + 1/2$ , level  $N$  with character  $\chi$  by  $M_{k+1/2}(N, \chi)$ , and the space of cusp forms of weight  $k + 1/2$ , level  $N$  with character  $\chi$  by  $S_{k+1/2}(N, \chi)$ . Then  $M_{k+1/2}(N, \chi)$  and  $S_{k+1/2}(N, \chi)$  are complex vector spaces.

For a modular form of half-integral weight

$$g(z) = \sum_{n=0}^{\infty} b(n)q^n \in M_{k+1/2}(N, \chi),$$

we define the action of Hecke operator  $T_{p^2}$  by

$$T_{p^2}(g)(z) = \sum_{n=0}^{\infty} b'(n)q^n,$$

where  $b'(n)$  are given by

$$b'(n) = b(p^2 n) + \chi(p) \left(\frac{-1}{p}\right)^k \left(\frac{n}{p}\right) p^{k-1} b(n) + \chi(p^2) p^{2k-1} b(n/p^2)$$

and  $b(n/p^2)$  are zero if  $p^2 \nmid n$ .

Now we give a short review of the theory of the Shimura correspondence. Let  $N$  be a positive integer which is divisible by four and  $\chi$  a Dirichlet character mod  $N$ . Then we define a vector space  $S_{3/2}^0(N, \chi)$  to be the subspace of  $S_{3/2}(N, \chi)$  generated by

$$\left\{f(z) = \sum_{n=1}^{\infty} \psi(n)nq^{tm^2} \mid N = 4 \text{ cond}(\psi)^2 t, \chi = \psi\chi_{-t} \text{ and } \psi(-1) = -1\right\}$$

and denote the orthogonal complement by  $S'_{3/2}(N, \chi)$ . Then we assume

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S'_{3/2}(N, \chi)$$

if  $k \geq 2$ , and

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S'_{3/2}(N, \chi)$$

if  $k = 1$ . Let  $t$  be a square-free positive integer. Define a number  $A_t(n)$  to be

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = \left( \sum_{n=1}^{\infty} \frac{\chi(n) \left(\frac{-1}{n}\right)^k \left(\frac{t}{n}\right)}{n^{s-k+1}} \right) \left( \sum_{n=1}^{\infty} \frac{b(tn^2)}{n^s} \right).$$

Then Shimura [14] proved that there is a positive integer  $M$  such that  $\mathrm{SH}_t(g(z)) = f_t(z) = \sum_{n=1}^{\infty} A_t(n)q^n \in S_{2k}(M, \chi^2)$ . (In fact, one can prove that  $M = N/2$ ). Furthermore for any  $t, t'$ , the difference between  $\mathrm{SH}_t(g)$  and  $\mathrm{SH}_{t'}$  is only constant multiple, so essentially this correspondence is independent of choice of  $t$ . This correspondence between modular forms is called the Shimura correspondence. Moreover if  $g$  is an eigenform for all Hecke operators  $T_{p^2}$  with  $(p, 2N) = 1$ , then the image of  $g$  under the Shimura correspondence is also an eigenform for all Hecke operators  $T_p$  with  $(p, 2N) = 1$  and the Hecke eigenvalue of  $T_{p^2}$  for  $g$  coincides with the Hecke eigenvalue for  $T_p$  for  $\mathrm{SH}_t(g)$ .

We recall the following result which is a useful version of Waldspurger's formula ([17, Théorém 1]) by Ono-Skinner. This formula gives a relation between the Fourier coefficients of modular forms of half-integral weight and the central values of twisted  $L$ -functions for modular forms.

**Theorem 2.1** (Ono-Skinner [9], (2a),(2b)). *Let  $f(z) = \sum_{n=1}^{\infty} a(n)q^n$  be a normalized newform of weight  $2k$ , level  $M$  with trivial character. Then there is  $\delta(f) \in \{\pm 1\}$ , a positive integer  $N$  with  $4M \mid N$ , a Dirichlet character  $\chi$  modulo  $N$ , a period  $\Omega_f \in \mathbb{C}^\times$  and a non-zero eigenform*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

with the property that  $g(z)$  maps to a twist of  $f$  under the Shimura correspondence and for all fundamental discriminant  $D$  with  $\delta(f)D > 0$  we have

$$b(D_0)^2 = \begin{cases} \alpha_D \frac{L(f \otimes \chi_D, k) D_0^{k-1/2}}{\Omega_f} & \text{if } (D, N) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\alpha_D$  and  $b(n)$  are algebraic integers in some finite extension of  $\mathbb{Q}$ . Moreover, there exists a finite set of primes  $S$  such that if  $D$  is a fundamental discriminant for which

- (1)  $\delta(f)D > 0$ ,
- (2)  $(D, N) = 1$ ,

then we have  $|L(f \otimes \chi_D, k) D_0^{k-1/2} / \Omega_f|_\lambda = |b(D_0)^2|_\lambda$  for  $\lambda \notin S$ .

### 3. SOME PROPERTIES OF FOURIER COEFFICIENTS OF MODULAR FORMS AND GALOIS REPRESENTATIONS

In this section we generalize some results of Serre [13] and Swinnerton-Dyer [16] using a result of Ribet [12]. These results should be well-known for specialists. However we give a short review for them, since it does not seem to be available in the literature. Let  $f = \sum_{n=1}^{\infty} a(n)q^n$  be a normalized newform of weight  $2k$  for  $\Gamma_0(N)$  with trivial character. Let  $E$  be the subfield of  $\mathbb{C}$  generated by the Fourier coefficients  $a(n)$  of  $f$ . Then  $E$  is a finite extension of  $\mathbb{Q}$ . Let  $\mathcal{O}_E$  be the ring of integers of  $E$ . For each prime  $\ell$ , we let  $\mathcal{O}_{E,\ell} = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$  and  $E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ .

**Theorem 3.1** (Deligne [3]). *For each prime  $\ell$ , there exists a continuous representation*

$$\rho_{f,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathcal{O}_{E,\ell}) \subset \text{GL}_2(E_\ell)$$

*unramified at all primes  $p \nmid N\ell$  such that  $\text{trace}\rho_{f,\ell}(\text{Frob}_p) = a(p)$  and  $\det\rho_{f,\ell}(\text{Frob}_p) = p^{2k-1}$  for all primes  $p \nmid N\ell$ , where  $\text{Frob}_p$  is the arithmetic Frobenius at  $p$ .*

For each prime  $\ell$ , denote

$$A_\ell = \left\{ g \in \text{GL}_2(\mathcal{O}_{E,\ell}) \mid \det(g) \in \mathbb{Z}_\ell^{\times(2k-1)} \right\},$$

where  $\mathbb{Z}_\ell^{\times(2k-1)}$  is the group of  $(2k-1)$ -th powers of elements in  $\mathbb{Z}_\ell^\times$ . Replacing  $\rho_{f,\ell}$  by an isomorphic representation, we may assume that for almost all  $\rho_{f,\ell}$  sends  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  to  $A_\ell$ . Then Ribet proved the following theorem.

**Theorem 3.2** (Ribet [12]). *Assume that  $f$  has no complex multiplication. Then for almost all  $\ell$ , we have  $\rho_{f,\ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) = A_\ell$ .*

We call the set of primes  $\ell$  with the property  $\rho_{f,\ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \neq A_\ell$  by the exceptional primes for  $f$ . Let  $S$  be the set of exceptional places for  $f$ . Let  $\varepsilon_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{Z}_\ell^\times$  be the  $\ell$ -adic cyclotomic character. Then by a similar argument with Swinnerton-Dyer [16], one can see that the image of

$$(\rho_{f,\ell}, \varepsilon_\ell) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_\ell^\times$$

is  $\{(g, \alpha) \in \text{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_\ell^\times \mid \det(g) = \alpha^{2k-1}\}$  if  $\ell$  is not exceptional. Since  $A_\ell$  contains an element with the form

$$\begin{pmatrix} \text{trace}\rho_{f,\ell}(\sigma) & -1 \\ \det\rho_{f,\ell}(\sigma) & 0 \end{pmatrix},$$

the map  $(\text{trace}\rho_{f,\ell}, \varepsilon_\ell) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathcal{O}_{E,\ell} \times \mathbb{Z}_\ell^\times$  is surjective. Moreover by a ramification argument, one can see that the map

$$\prod_{\ell \notin S} (\text{trace}\rho_{f,\ell}, \varepsilon_\ell) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \prod_{\ell \notin S} (\mathcal{O}_{E,\ell} \times \mathbb{Z}_\ell^\times)$$

is also surjective. Therefore we have the following result which is a generalization of a result of Serre [13, THÉORÈME 11] using Chebotarev density theorem.

**Theorem 3.3.** *Assume that  $f$  has no complex multiplication. Let  $t$  be a positive integer and  $\alpha$  a non-zero integer in  $E$ . Fix  $\beta \in \mathcal{O}_E/\alpha\mathcal{O}_E$  and  $r \in (\mathbb{Z}/t\mathbb{Z})^\times$ . Suppose that  $\alpha$  does not contain a prime divisor which divides an exceptional prime for  $f$ . Then the set of prime  $p$  with the properties  $a(p) \equiv \beta \pmod{\alpha}$  and  $p \equiv r \pmod{t}$  has positive density.*

#### 4. INDIVISIBILITY OF FOURIER COEFFICIENTS OF MODULAR FORMS OF HALF-INTEGRAL WEIGHT

In this section, we give a result on modulo  $\ell$  indivisibility of Fourier coefficients of half-integral weight modular forms using a method of Kohnen-Ono [7] and James-Ono [5]. Our result is a refinement of a result of Bruinier [2] and Ono-Skinner [10].

To consider the indivisibility of Fourier coefficients of half-integral weight modular forms, we will use the following results.

**Theorem 4.1** (Sturm [15]). *Let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in M_k(N, \chi)$$

be a half-integral or integral weight modular form for which the coefficients  $b(m)$  are algebraic integers contained in a number field  $E$ . Let  $v$  be a finite place of  $E$  and let

$$\text{ord}_v(g) = \begin{cases} +\infty & \text{if } b(n) \equiv 0 \pmod{v} \text{ for all } n, \\ \min\{n \mid b(n) \not\equiv 0 \pmod{v}\} & \text{otherwise} \end{cases}$$

Moreover put

$$\mu = \frac{k}{12} [\Gamma_0(1) : \Gamma_0(N)] = \frac{kN}{12} \prod_{p|N} \frac{p+1}{p}.$$

Assume that

$$\text{ord}_v(g) > \mu,$$

then  $\text{ord}_v(g) = +\infty$ .

**Remark 4.2** (cf. [5] Proposition 5). In [15], Sturm proved this theorem for integral weight modular forms with trivial character, but the general case follows by taking an appropriate power of  $g$ .

**Lemma 4.3** (Shimura, [14] Section 1). Suppose

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

is a half integral weight cusp form and  $p$  is a prime. We define  $(U_p g)(z)$ ,  $(V_p g)(z)$  by

$$\begin{aligned} (U_p g)(z) &= \sum_{n=1}^{\infty} u_p(n)q^n = \sum_{n=1}^{\infty} b(pn)q^n, \\ (V_p g)(z) &= \sum_{n=1}^{\infty} v_p(n)q^n = \sum_{n=1}^{\infty} b(n)q^{pn}. \end{aligned}$$

Then

$$(U_p g)(z), (V_p g)(z) \in S_{k+1/2} \left( Np, \chi \left( \frac{4p}{\cdot} \right) \right).$$

Let

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n \in M_k(N, \chi)$$

be an integral weight modular form for which the coefficients  $a(m)$  are algebraic integers in  $E$ . For a prime  $\lambda$  of  $E$  and positive integers  $r, t$  with  $(r, t) = 1$ , define  $T(r, t)$  and  $T(\lambda, r, t)$  by

$$T(r, t) = \{p : \text{prime} \mid a(p) = 0, p \equiv r \pmod{t}\}$$

and

$$T(\lambda, r, t) = \{p : \text{prime} \mid a(p) \equiv 0 \pmod{\lambda}, p \equiv r \pmod{t}\}.$$

For a positive real number  $X$ , we also denote  $T(r, t, X) = \{p \in T(r, t) \mid p \leq X\}$  and  $T(\lambda, r, t, X) = \{p \in T(\lambda, r, t) \mid p \leq X\}$ .

For  $g = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi) \cap \mathcal{O}_{E, \lambda}[[q]]$ , denote  $s_{\lambda}(g) = \min\{\text{ord}_{\lambda}(b(n)) \mid n \in \mathbb{Z}_{>0}\}$ . The following two lemmas give an estimate for indivisibility of Fourier coefficients of modular forms of half integral weight.

**Lemma 4.4.** *Let  $\ell$  be a prime greater than 3. Let  $f(z) = \sum_{n=1}^{\infty} a(n)q^n$  be a normalized Hecke eigen newform of weight  $2k$ , level  $M$  with trivial character and let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

*be the eigenform given in Theorem 2.1. Assume that  $f$  has complex multiplication in the sense of Ribet [11] and  $\lambda$  be a prime in  $E$  above  $\ell$ . If there exists an integer  $D'$  such that  $\delta(f)D' > 0$ ,  $(D', N) = 1$ ,  $\varepsilon = \left(\frac{D'}{\ell}\right) \neq 0$  and  $\text{ord}_{\lambda}(b(|D'|)) = s_{\lambda}(g)$ , then*

$$\# \left\{ D \in S(X) \mid \left(\frac{D}{\ell}\right) = \varepsilon, \text{ord}_{\lambda}(b(D)) = s_{\lambda}(g) \right\} \gg_{f, \ell} \frac{\sqrt{X}}{\log X}.$$

*Proof.* By dividing  $g$  by  $\lambda^{s_{\lambda}(g)}$ , we may assume  $s_{\lambda}(g) = 0$ . If we put

$$b_0(n) = \begin{cases} b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left(\frac{n}{\ell}\right) = \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$g_0(z) = \sum_{n=1}^{\infty} b_0(n)q^n \in S_{k+1/2}(N\ell^2, \chi')$$

for a suitable character  $\chi'$ . Since  $f$  has complex multiplication, so there exists a imaginary quadratic field  $K$  such that for every prime  $p$  satisfying  $p \equiv 3 \pmod{4}$ ,  $(p, N) = 1$  and  $\left(\frac{\Delta_K}{p}\right) = -1$  we have  $a(p) = 0$ , where  $\Delta_K$  is the discriminant of  $K$ . Therefore, for such  $p$ , using the formulae for the action of Hecke operator  $T_{p^2}$ , we find that

$$b(p^2n) + \chi'(p)p^{k-1} \left(\frac{(-1)^k n}{p}\right) b(n) + \chi'(p^2)p^{2k-1}b(n/p^2) = 0.$$

Hence if  $(r, t) = 1$ ,  $4 \mid t$ ,  $r \equiv 3 \pmod{4}$ , then

$$\#T(r, t, X) = \#\{p \in T(r, t) \mid p \leq X\} \gg_f \frac{X}{\log X}$$

and for any  $p \in T(r, t)$  we have

$$(4.1) \quad b(p^2n) = -\chi'(p)p^{k-1} \left(\frac{(-1)^k n}{p}\right) b(n) - \chi'^2(p)p^{2k-1}b(n/p^2).$$

Put  $\kappa = (k + \frac{1}{2}) \frac{[\Gamma_0(1) : \Gamma_0(N\ell^2)]}{12} + 1$ . Now, we choose  $(r_0, t_0)$  satisfying the following properties:

- (1)  $N\ell^2 \mid t_0$ ,  $(r_0, t_0) = 1$ ,  $\chi'(r_0) = 1$  and  $p \equiv 3 \pmod{4}$ .
- (2) If  $p$  is a prime with  $p \equiv r_0 \pmod{t_0}$ , then  $\left(\frac{(-1)^k n}{p}\right) = -1$  for any  $1 \leq n \leq \kappa$  with  $(n, N\ell^2) = 1$ .
- (3) For each prime  $p \equiv r_0 \pmod{t_0}$  we have  $\left(\frac{\Delta_K}{p}\right) = -1$ .
- (4) Each prime  $p \equiv r_0 \pmod{t_0}$  satisfies  $\left| \chi'(p^2)p - \chi'(p) \left(\frac{(-1)^k |D'|}{p}\right) \right|_{\lambda} = 1$ .

If  $p \in T(r_0, t_0)$  is a sufficiently large prime, for all  $1 \leq n \leq \kappa$

$$u_p(pn) = b_0(p^2n) = -\chi'(p)p^{k-1} \left( \frac{(-1)^k n}{p} \right) b_0(n) - p^{2k-1} \chi'^2(p) b_0(n/p^2)$$

Since  $b_0(n/p^2) = 0$ , we have  $u_p(pn) = \chi'(p)p^{k-1}b_0(n) = p^{k-1}b_0(p) = p^{k-1}v_p(pn)$ . By the relation (4.1),

$$v_p(p^3|D'|) = b_0(p^2|D'|) = -\chi'(p)p^{k-1} \left( \frac{(-1)^k |D'|}{p} \right) b_0(|D'|),$$

and

$$u_p(p^3|D'|) = b_0(p^4|D'|) = -p^{2k-1} \chi'(p^2) b_0(|D'|).$$

Therefore by the assumption and the choice of  $(r_0, t_0)$ ,

$$|u_p(p^3|D'|) - p^{k-1}v_p(p^3|D')|_\lambda = \left| \left( \chi'(p^2)p^{2k-1} - \chi'(p)p^{2k-2} \left( \frac{(-1)^k |D'|}{p} \right) \right) b_0(|D'|) \right|_\lambda = 1.$$

Hence

$$\text{ord}_\lambda(U_p g_0 - p^{k-1} V_p g_0) < +\infty.$$

By Theorem 4.1 and Lemma 4.3, there exists a integer  $n_p$  such that

$$1 \leq n_p \leq \left( k + \frac{1}{2} \right) \frac{[\Gamma_0(1) : \Gamma_0(N\ell^2 p)]}{12} = \kappa(p+1), (n_p, p) = 1$$

and

$$b_0(n_p p) = u_p(n_p) \not\equiv p^{k-1} v_p(n_p) = 0 \pmod{\lambda}.$$

Consequently, let  $D_{\text{sf}}$  be the square-free part of  $D = n_p p$ , then

$$|b_0(D_{\text{sf}})|_\lambda = 1.$$

For convenience, let  $p_i$  be the primes in  $T(r_0, t_0)$  in increasing order, and let  $D_i$  be the square-free part of  $p_i n_{p_i}$ . If  $r < s < t$  and  $D_r = D_s = D_t$ , then  $p_r p_s p_t | D_r$ . However this can only occur for finitely many  $r, s$  and  $t$  since  $|D_i| < \kappa p_i (p_i + 1)$ . Therefore, the number of distinct  $|D_i| < X$  is at least half the number of  $p \in T(r_0, t_0)$  with  $p \leq \sqrt{X/\kappa}$ . Therefore the lemma follows from  $\#T(r_0, t_0, X) \gg_{f,\lambda} X/\log X$ .  $\square$

**Lemma 4.5.** *Let  $f(z) = \sum_{n=1}^{\infty} a(n)q^n$  be a normalized Hecke eigen newform of weight  $2k$ , level  $M$  with trivial character. Denote  $E = \mathbb{Q}(\{a(n)|n \geq 1\})$  and let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

*be the eigenform given in Theorem 2.1. We fix a prime number  $\ell$  greater than 3 and let  $\lambda$  be a prime in  $E$  above  $\ell$ . Assume that  $f$  does not have complex multiplication and the image of the Galois representation associated to  $f$*

$$\rho_{f,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathcal{O}_{E,\ell})$$

*coincides with  $A_\ell$ . If there exists an integer  $D'$  such that  $\delta(f)D' > 0$ ,  $(D', N) = 1$ ,*

*$\varepsilon = \left( \frac{D'}{\ell} \right) \neq 0$  and  $\text{ord}_\lambda(b(|D'|)) = s_\lambda(g)$ , then*

$$\# \left\{ D \in S(X) \mid \left( \frac{D}{\ell} \right) = \varepsilon, \text{ord}_\lambda(b(D)) = s_\lambda(g) \right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$



*Proof.* First, we may assume  $\text{ord}_\lambda(g) = 0$ . If we put

$$b_0(n) = \begin{cases} b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left(\frac{n}{\ell}\right) = \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$g_0(z) = \sum_{n=1}^{\infty} b_0(n)q^n \in S_{k+1/2}(N\ell^2, \chi')$$

for a suitable character  $\chi'$ . If  $a(p) \equiv 0 \pmod{\lambda}$ , by the formula for the action of Hecke operator  $T_{p^2}$  we find that

$$b(p^2n) + \chi'(p)p^{k-1} \left(\frac{(-1)^kn}{p}\right) b(n) + \chi'^2(p)p^{2k-1}b(n/p^2) \equiv 0 \pmod{\lambda}.$$

By the assumption,  $\ell$  is not exceptional. Hence Theorem 3.3 implies

$$\#T(\lambda, r, t, X) = \#\{p \in T(\lambda, r, t) \mid p \leq X\} \gg_{f, \lambda} \frac{X}{\log X}$$

and for each  $p \in T(\lambda, r, t)$

$$(4.2) \quad b(p^2n) \equiv -\chi'(p)p^{k-1} \left(\frac{(-1)^kn}{p}\right) b(n) - \chi'^2(p)p^{2k-1}b(n/p^2) \pmod{\lambda}.$$

Let  $\kappa$  be the number as in the proof of Lemma 4.4. Now, we choose  $(r_0, t_0)$  satisfying the following properties:

- (1)  $N\ell^2 \mid t_0$ ,  $(r_0, t_0) = 1$ ,  $\chi'(r_0) = 1$ .
- (2) If  $p$  is a prime with  $p \equiv r_0 \pmod{t_0}$ , then  $\left(\frac{(-1)^kn}{p}\right) = -1$  for any  $1 \leq n \leq \kappa$  with  $(n, N\ell^2) = 1$ .
- (3) For each prime  $p \equiv r_0 \pmod{t_0}$  we have  $\left(\frac{(-1)^k|D'|}{p}\right) = -1$ .
- (4) Each prime  $p \equiv r_0 \pmod{t_0}$  has the property that  $1 + p \not\equiv 0 \pmod{\lambda}$ .

If  $p \in T(\lambda, r_0, t_0)$  is a sufficiently large prime, for all  $1 \leq n \leq \kappa$  with  $(n, N\ell^2) = 1$ , one has

$$u_p(pn) = b_0(p^2n) \equiv -p^{k-1} \left(\frac{(-1)^kn}{p}\right) b_0(n) - p^{2k-1}b_0(n/p^2) = p^{k-1}b_0(n) = p^{k-1}v_p(pn) \pmod{\lambda}.$$

By the relation (4.2), we have

$$v_p(p^3|D'|) = b_0(p^2|D'|) \equiv p^{k-1}b_0(|D'|) \pmod{\lambda},$$

also

$$u_p(p^3|D'|) = b_0(p^4|D'|) \equiv -p^{2k-1}b_0(|D'|) \pmod{\lambda}.$$

Therefore by assumption and the choice of  $(r_0, t_0)$ ,

$$p^{k-1}v_p(p^3|D'|) - u_p(p^3|D'|) \equiv p^{2k-2}(1+p)b_0(|D'|) \not\equiv 0 \pmod{\lambda}.$$

Hence

$$\text{ord}_\lambda(U_p g_0 - p^{k-1}V_p g_0) < +\infty.$$

By Theorem 4.1 and Lemma 4.3, there exists a integer  $n_p$  such that

$$1 \leq n_p \leq (k+1/2)[\Gamma_0(1) : \Gamma_0(N\ell^2 p)]/12 = \kappa(p+1), (n_p, p) = 1$$

and

$$b_0(n_p p) = u_p(n_p) \not\equiv p^{k-1} v_p(n_p) = 0 \pmod{\lambda}.$$

In particular, let  $D_{\text{sf}}$  be the square-free part of  $D = n_p p$ , then

$$|b_0(D_{\text{sq}})|_{\lambda} = 1.$$

Now the lemma follows from the same argument with the proof of the previous lemma using Theorem 3.3.  $\square$

### Proof of Theorem 1.1.

Now we give the proof of Theorem 1.1. Let

$$g(z) = \sum_{n=1}^{\infty} b(n) q^n \in S_{k+1/2}(N, \chi)$$

be the eigenform given in Theorem 2.1 for  $f$ .

By replacing  $f$  by a suitable quadratic twist of  $f$  if necessary, we may assume that  $\varepsilon = \delta(f)$ , where  $\varepsilon$  is the sign of the functional equation of  $L(f, s)$ . By the result of Friedberg and Hoffstein [4], we can take an integer  $D'$  such that  $\delta(f)D' > 0$ ,  $(D', 2N) = 1$  and  $b(D') \neq 0$ . In particular, for almost all finite places  $\lambda$  of  $E$  we have

$$|b(D')|_{\lambda} = 1.$$

Thus by Lemmas 4.4, 4.5, Theorem 2.1 and Theorem 3.3, for all but finitely many primes  $\lambda$  we have

$$\# \left\{ D \in S(X) \mid \delta(f) \cdot D > 0, (\ell, D) = 1 \text{ and } \left| \frac{L(f \otimes \chi_D, k) D_0^{k-1/2}}{\Omega_f} \right|_{\lambda} = 1 \right\} \gg_{f, \lambda} \frac{\sqrt{X}}{\log X}.$$

This completes the proof.

## 5. INDIVISIBILITY FOR THE NON-CENTRAL CRITICAL VALUES

In this section, we consider a special case for non-central values of  $L$ -functions for modular forms. We fix a prime  $\ell$  greater than 7 and let  $f = \sum_{n=1}^{\infty} a(n) q^n$  be a normalized Hecke eigenform of weight  $\ell + 1$  for  $\text{SL}_2(\mathbb{Z})$ . Let  $\lambda$  be a prime in a number field  $E$ . We assume that the integer ring of  $E$  contains all Fourier coefficients of  $f$  and choose a period  $\Omega_f^{\pm}$  as in Ash-Stevens [1, Theorem 4.5]. Then for any Dirichlet character  $\chi$ , the quotient  $\tau(\chi^{-1}) \frac{L(f \otimes \chi, 1)}{(2\pi i) \Omega_f^{\pm}}$  is an integer in  $E_{\lambda}(\chi)$  where  $\tau$  is the Gauss sum and  $\pm = \chi(-1)$ .

**Theorem 5.1.** *Let  $\lambda$  be a prime in  $E$  above  $\ell$ . We assume the following conditions.*

- (1) *There exists a unique eigenform  $F$  of weight 2 for  $\Gamma_0(\ell)$  such that*

$$F \equiv f \pmod{\lambda}.$$

- (2)  *$\ell$  is not exceptional.*

- (3) *There exists an square-free negative integer  $d_0$  such that  $(d_0, 2\ell) = 1$ ,  $\chi_{d_0}(\ell) = -\varepsilon(F)$ , where  $\varepsilon(F)$  is the sign of functional equation of  $L(F, s)$  and*

$$\frac{L(f \otimes \chi_{d_0}, 1) \sqrt{d_0}}{(2\pi i) \Omega_f^{\pm}} \not\equiv 0 \pmod{\lambda}.$$

Then we have

$$\# \left\{ D \in S(X) \mid \frac{L(f \otimes \chi_D, 1) \sqrt{D}}{(2\pi i) \Omega_f^{\pm}} \not\equiv 0 \pmod{\lambda} \right\} \gg_{f, \lambda} \frac{\sqrt{X}}{\log X}.$$

For the proof, we recall a result of Ash and Stevens.

**Theorem 5.2** (Ash-Stevens, [1]). *Let  $k$  be a positive integer less than  $\ell + 2$  and  $f = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(1))$  an eigenform satisfying the assumptions of Theorem 5.1. We fix a prime  $\lambda$  above  $\ell$  in a number field  $E$  which contains all Fourier coefficients of  $f$ . Assume that*

- (1) *There exists a prime  $q$  satisfying  $a(q) \not\equiv q^{k-1} + 1 \pmod{\lambda}$ .*
- (2) *There exists a unique eigenform  $F \in S_2(\Gamma_1(\ell))$  such that  $f \equiv F \pmod{\lambda}$ .*

*Then there exists a complex number  $\Omega_F^{\pm}$  such that for any Dirichlet character  $\chi$  satisfying  $(\text{cond } \chi, p) = 1$ , we have*

$$\frac{\tau(\chi^{-1})L(f \otimes \chi, 1)}{(2\pi i)\Omega_f^{\pm}} \equiv \frac{\tau(\chi^{-1})L(F \otimes \chi, 1)}{(2\pi i)\Omega_F^{\pm}} \pmod{\lambda}.$$

Now we prove Theorem 5.1. By the Kohnen-Zagier formula [6], there exists an eigenform

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{3/2}(\Gamma_0(4\ell))$$

such that for any negative square-free integer  $D$  satisfying  $\left(\frac{D}{\ell}\right) = -\varepsilon(F)$ ,

$$|b(|D|)|^2 = 2 \cdot \frac{\sqrt{D}}{\pi} \cdot \frac{\langle g, g \rangle}{\langle F, F \rangle} L(F \otimes \chi_D, 1),$$

where  $\langle \cdot, \cdot \rangle$  is the Petersson inner product. We can normalize  $g$  by the relation  $\frac{\langle F, F \rangle}{\langle g, g \rangle} = \Omega_f^{\pm}$ . Taking a linear combination of twists of  $g$ , one may assume  $b(|D|) = 0$  if  $\left(\frac{D}{\ell}\right) \neq -\varepsilon(F)$  and  $D < 0$ . From the assumptions of the theorem,  $\ell$  is not exceptional. This implies the existence of a prime  $q$  satisfying  $a(q) \not\equiv q^{k-1} + 1 \pmod{\lambda}$ , therefore the assumptions of Theorem 5.1 implies the assumptions of Theorem 5.2. Since  $\tau(\chi_D)^{-1} = \pm 1/\sqrt{D}$ , one can see that

$$\frac{L(f \otimes \chi, 1)\sqrt{D}}{(2\pi i)\Omega_f^{\pm}} \equiv \frac{L(F \otimes \chi, 1)\sqrt{D}}{(2\pi i)\Omega_F^{\pm}} = |b(|D|)|^2 \cdot c \pmod{\lambda}$$

with a  $\lambda$ -adic unit  $c$ . By the assumption (3), we have

$$\text{ord}_{\lambda} \left( \frac{L(f \otimes \chi_{d_0}, 1)\sqrt{d_0}}{(2\pi i)\Omega_f^{\pm}} \right) = 0,$$

therefore  $\text{ord}_{\lambda}(b(d_0)) = \min\{\text{ord}_{\lambda}(b(n)) \mid n : \text{square-free}, \chi_{d_0}(\ell) = -\varepsilon(f)\}$ . Hence Lemma 4.5 implies

$$\#\{D \in S(X) \mid \chi_D(\ell) = -\varepsilon(f), \text{ord}_{\lambda}(b(D)) = s\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X},$$

thus we have

$$\#\left\{D \in S(X) \mid \frac{L(f \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)\Omega_f^{\pm}} \not\equiv 0 \pmod{\lambda}\right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

This completes the proof.

**Remark 5.3.** *Lemma 4.5 states only for  $g$  given in Theorem 2.1, but one can show the similar result for any eigenform  $g \in S_{k+1/2}(N, \chi)$  if  $k \geq 2$  ( $S'_{\frac{3}{2}}(N, \chi)$  if  $k = 1$ ) corresponding to some eigenform  $f \in S_{2k}(\Gamma_0(M))$  under the Shimura correspondence.*

**Example 5.4.** *Let*

$$f = \Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \in S_{12}(\Gamma_0(1))$$

and

$$F = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 \in S_2(\Gamma_0(11)).$$

Then it is well-known that  $f \equiv F \pmod{11}$ ,  $\dim S_2(\Gamma_0(11)) = 1$  and the mod 11 Galois representation associated to  $f$  is surjective. Moreover one can check that

$$\frac{L(\Delta \otimes \chi_{-3}, 1)}{\Omega_{\Delta \otimes \chi_{-3}}^+} = 36741600 \not\equiv 0 \pmod{11}$$

by using MAGMA. So the assumptions of Theorem 5.1 are satisfied for  $f = \Delta$ . Hence we have

$$\# \left\{ D \in S(X) \mid \frac{L(\Delta \otimes \chi_D, 1) \sqrt{D}}{(2\pi i) \Omega_{\Delta}^{\pm}} \not\equiv 0 \pmod{11} \right\} \gg \frac{\sqrt{X}}{\log X}.$$

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